Probability and Measure Lecture Notes (2024/2025)

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Contents

Definition *Riemann integral*

Let $f : [a, b] \in \mathbb{R}$ be a bounded function, and define:

$$\Pi = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n\} \qquad |\Pi| = \max\{x_{i+1} - x_i\}$$

The **Riemann integral** is defined by:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{|\Pi|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \qquad \xi_i \in (x_i, x_{i-1})$$

1 Measure spaces

1.1 Algebras and measures

1.1.1 Finitely additive algebras

Definition *Algebra*

A collection $\mathcal{A}$ of subsets of a set Ω is a (finitely additive) **algebra** if:

1. $\Omega \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
3. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

Lemma

Let $\mathcal{A}$ be an algebra on Ω and $A, B \in \mathcal{A}$. Then the following are also in $\mathcal{A}$:

$$A \cap B \qquad A \setminus B \qquad A \Delta B \text{ (symmetric difference)} \qquad \varnothing$$

Definition *Finitely additive measure*

A **finitely additive measure** μ on an algebra $\mathcal{A}$ is a function $\mu : A \to [0, \infty]$ which satisfies

1. $\mu(\varnothing) = 0$
2. If $A, B \in \mathcal{A}$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$

Lemma

Let μ be a finitely additive measure on an algebra $\mathcal{A}$. Then

1. $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.
2. $A \subset B \implies \mu(B) = \mu(A) + \mu(B \setminus A)$
3. $A \subset B$ and $\mu(A) \leq \infty \implies \mu(B) - \mu(A) = \mu(B \setminus A)$

Lemma *Construction of pairwise disjoint sets*

Let $A_n \in \Omega, n \in \mathbb{N}$ and define the subsets

$$A'_1 = A_1 \qquad A'_n = A_n \setminus (A_1 \cup \ldots \cup A_{n-1}) \qquad n \geq 2$$

Then the sets A'_n are pairwise disjoint and

$$\bigcup_{k=1}^{n} A'_k = \bigcup_{k=1}^{n} A_k \qquad \bigcup_{n=1}^{\infty} A'_n = \bigcup_{n=1}^{\infty} A_n$$

Note: if A_n are monotonically non-decreasing ($A_{n-1} \subseteq A_n$), then

$$A'_n = A_n \setminus A_{n-1} \qquad \bigcup_{k=1}^{n} A'_k = A_n \qquad \bigcup_{n=1}^{\infty} A'_n = \bigcup_{n=1}^{\infty} A_n$$

1.1.2 σ-algebras and measures

Definition *σ-algebra*

A collection $\mathcal{A}$ of subsets of a set Ω is a σ-**algebra** if

1. $\Omega \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
3. $A_n \in \mathcal{A}$ for all $n \in \mathbb{N} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Lemma

If $\mathcal{A}$ is a σ-algebra, then

$$A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \implies \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

Definition *Measure*

A **measure** μ on a σ-algebra $\mathcal{A}$ is a function $\mu : A \to [0, \infty]$ which satisfies

1. $\mu(\varnothing) = 0$
2. If $A_n \in \mathcal{A}$ are pairwise disjoint, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Proposition *Countable subadditivity*

Let μ be a measure on a σ-algebra $\mathcal{A}$.

$$A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Corollary

The countable union of sets of measure 0 is a set of measure 0.

Definition *Measure space*

A **measure space** is a triple $(\omega, \mathcal{A}, \mu)$ consisting of a set Ω, a σ-algebra $\mathcal{A}$ on Ω, and a measure $\mu : \mathcal{A} \to [0, \infty]$.
A measure space $(\omega, \mathcal{A}, \mu)$ is a **finite measure space** if $\mu(\Omega) < \infty$ and a **probability space** if $\mu(\Omega) = 1$.

1.2 Monotone convergence

Definition *Monotone convergence*

$$A_n \uparrow A \iff A_1 \subseteq A_2 \subseteq \ldots \text{ and } A = \bigcup A_i$$

$$A_n \downarrow A \iff A_1 \supseteq A_2 \supseteq \ldots \text{ and } A = \bigcap A_i$$

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $A_n \in \mathcal{A}$. Then

1. if $A_n \uparrow A$, then $A \in \mathcal{A}$ and $\mu(A_n) \uparrow \mu(A)$
2. if $A_n \downarrow A$ and $\mu(A_1) < \infty$, then $A \in \mathcal{A}$ and $\mu(A_n) \downarrow \mu(A)$

Theorem

If $\mathcal{A}$ is a σ-algebra and μ a finitely additive measure on Ω such that

$$\mu(\Omega) < \infty \quad \text{and} \quad A_n \downarrow \varnothing \implies \mu(A_n) \downarrow 0$$

then μ is a measure.

1.2.1 σ-finite measures

Lemma

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. The following are equivalent:

1. There exist $\Omega_n \in \mathcal{A}$ such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup \Omega_n$
2. There exist $\Omega_n \in \mathcal{A}$ such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup \Omega_n$, and Ω_n are mutually disjoint.
3. There exist $\Omega_n \in \mathcal{A}$ such that $\mu(\Omega_n) < \infty$ and $\Omega_n \uparrow \Omega$

Definition *σ-finite measure*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If Ω_n as in the previous lemma exist, then $(\Omega, \mathcal{A}, \mu)$ is **σ-finite**.

Proposition

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $\Omega_n \in \mathcal{A}$ be such that $\mu(\Omega_n) < \infty$ and $\Omega_n \uparrow \Omega$. Then

$$\mu_n(A) := \mu(A \cap \Omega_n) \qquad A \in \mathcal{A}$$

defines a finite measure on Ω, and $\mu_n(A) \uparrow \mu(A)$ for all $A \in \mathcal{A}$.

1.3 Generators of σ-algebras

Lemma

The intersection of a non-empty family of σ-algebras on Ω is a σ-algebra.

Proposition

Let $\mathcal{E}$ be a family of subsets of Ω. Then there exists a unique σ-algebra $\mathcal{A}$ such that

1. $\mathcal{E} \subset \mathcal{A}$
2. If $\mathcal{B}$ is a σ-algebra containing $\mathcal{E}$, then $\mathcal{A} \subset \mathcal{B}$

Definition *Generated σ-algebra*

$\mathcal{A}$ as in the previous proposition is the σ-algebra **generated** by the **generator** $\mathcal{E}$, denoted $\sigma(\mathcal{E})$.

Definition *Borel σ-algebra*

Let Ω be a topological space. The σ-algebra generated by the open sets is called the **Borel σ-algebra** $\mathcal{B}(\Omega)$. The elements of a Borel σ-algebra are called **Borel sets**.

Proposition

The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by:

1. the collection of closed subsets of $\mathbb{R}$
2. the collection of intervals $(-\infty, c]$, $c \in \mathbb{R}$
3. the collection of intervals $(a, b]$, $a < b$

The Borel σ-algebra $\mathcal{B}^d$ on $\mathbb{R}^d$ is generated by:

1. the collection of closed subsets of $\mathbb{R}^d$
2. the collection of half-spaces $\{(x_1, \dots, x_d) : x_i \leq b\}$, where $b \in \mathbb{R}$ and $1 \leq i \leq d$
3. the collection of rectangles $(a_1, b_1] \times \dots \times (a_d, b_d]$ where $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, and $1 \leq i \leq d$
4. the collection of compact subsets of $\mathbb{R}^d$
5. the collection of rectangles $[a_1, b_1] \times \dots \times [a_d, b_d]$ where $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, and $1 \leq i \leq d$

1.4 Dynkin systems

Definition *Dynkin system*

A collection $\mathcal{A}$ of subsets of a set Ω is a **Dynkin system** if

1. $\Omega \in \mathcal{D}$
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
3. $A_n \in \mathcal{A}$ and A_n are pairwise disjoint for all $n \in \mathbb{N} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Lemma

The intersection of a non-empty family of Dynkin systems on Ω is a Dynkin system.

Proposition

Let $\mathcal{E}$ be a family of subsets of Ω. Then there exists a unique Dynkin system $\mathcal{D}$ such that

1. $\mathcal{E} \subset \mathcal{D}$
2. If $\mathcal{F}$ is a Dynkin system containing $\mathcal{E}$, then $\mathcal{D} \subset \mathcal{F}$

Definition *Generated Dynkin system*

$\mathcal{D}$ as in the previous proposition is the Dynkin system **generated** by the **generator** $\mathcal{E}$, denoted $d(\mathcal{E})$.

Lemma

$$d(\mathcal{E}) \subset \sigma(\mathcal{E})$$

Lemma

$$\mathcal{D} \text{ is a } \sigma\text{-algebra} \iff \mathcal{D} \text{ is a Dynkin system which is closed under intersection}$$

Lemma

Let $\mathcal{E}$ be a collection of subsets of Ω, let $D \in d(\mathcal{E})$ and define:

$$\mathcal{D}_D := \{A \in \mathcal{P}(\Omega) : A \cap D \in d(\mathcal{E})\}$$

1. $\mathcal{D}_D$ is a Dynkin system.
2. If $\mathcal{E}$ is closed under intersection, then $d(\mathcal{E}) \subset \mathcal{D}_D$.
3. If $\mathcal{E}$ is closed under intersection, then $d(\mathcal{E})$ is closed under intersection.

Theorem

Let $\mathcal{E}$ be a collection of subsets of Ω. If $\mathcal{E}$ is closed under intersections, then $d(\mathcal{E}) = \sigma(\mathcal{E})$

Corollary

Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{E}$ a generator of $\mathcal{A}$ which is closed under intersection.
Let μ and ν be finite measures on $\mathcal{A}$ such that:

1. $\mu(A) = \nu(A)$ for all $A \in \mathcal{E}$
2. $\mu(\Omega) = \nu(\Omega)$

Then $\mu = \nu$.

Corollary

Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{E}$ a generator of $\mathcal{A}$ which is closed under intersection.
Let μ and ν be σ-finite measures on $\mathcal{A}$ such that:

1. $\mu(A) = \nu(A)$ for all $A \in \mathcal{E}$
2. $\mu(\Omega_n) = \nu(\Omega_n) < \infty$ for some sequence $\Omega_n \in \mathcal{E}$ such that $\Omega_n \uparrow \Omega$

Then $\mu = \nu$.

1.5 Completion

Definition *Complete measure space*

A measure space $(\Omega, \mathcal{A}, \mu)$ is **complete** if

$$A \subset B \in \mathcal{A} \text{ and } \mu(B) = 0 \implies A \in \mathcal{A}$$

Definition *Completion*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Define:

$$\mathcal{Z} := \{N \subset \Omega : \text{ there exists } F \in \mathcal{A} \text{ such that } N \subset F \text{ and } \mu(F) = 0\}$$

$$\overline{\mathcal{A}} := \{E \cup N : E \in \mathcal{A}, N \in \mathcal{Z}\}$$

Then we define the **completion** $\overline{\mu}$ of μ on $\overline{\mathcal{A}}$ as

$$\overline{\mu}(E \cup N) = \mu(E) \qquad E \in \mathcal{A}, N \in \mathcal{Z}$$

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then the triple $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space extending $(\Omega, \mathcal{A}, \mu)$. Moreover, this extension is minimal: if $(\Omega, \mathcal{A}', \mu')$ is another complete measure space which extends μ, then it also extends $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$.

Corollary

The completion of a σ-finite measure space is σ-finite.

Proposition

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\Omega' \subset \Omega$. Then the **trace** of $\mathcal{A}$ on Ω' defined by

$$\mathcal{A}' = \{A \cap \Omega' : A \in \mathcal{A}\}$$

is a σ-algebra on Ω', and if $\Omega' \in \mathcal{A}$ then

$$u'(B) := \mu(B) \qquad B \in \mathcal{A}'$$

defines a measure on Ω'. If $(\Omega, \mathcal{A}, \mu)$ is complete, then so is $(\Omega', \mathcal{A}', \mu')$.

2 Construction of measures

2.1 Outer measures

Definition *Outer measure*

An extended real-valued function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ is an **outer measure** on Ω if

1. $\mu^*(\varnothing) = 0$
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$
3. $A_n \subset \Omega, n \in \mathbb{N} \implies \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

A set $A \subset \Omega$ is **measurable** with respect to μ^* if

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \cap A^c) \qquad \text{for all } Z \subset \Omega$$

Lemma

Let μ^* be an outer measure on Ω. A set $A \subset \Omega$ is measurable if and only if

$$Z \subset \Omega \text{ and } \mu^*(Z) < \infty \quad \implies \quad \mu^*(Z) \geq \mu^*(Z \cap A) + \mu^*(Z \cap A^c)$$

Lemma

Let $A \subset \Omega$ satisfy $\mu^*(A) = 0$. Then A is measurable with respect to μ^*

Theorem

Let μ^* be an outer measure on Ω and let $\mathcal{A}^*$ be the set of all μ^*-measurable sets $A \subset \Omega$.

1. $\mathcal{A}^*$ is a σ-algebra on Ω.
2. The restriction of μ^* to $\mathcal{A}^*$ is a complete measure.

2.2 Lebesgue measure

2.2.1 Lebesgue outer measure

Definition *Closed d-dimensional rectangle*

A **closed d-dimensional rectangle** is a subset of $\mathbb{R}^d$ of the form

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d] \qquad a_i, b_i \in \mathbb{R} \quad a_i \leq b_i \quad i = 1, \ldots, d$$

Its volume is defined by

$$\ell(R) = (b_1 - a_1) \cdots (b_d - a_d)$$

Definition *Lebesgue outer measure*

Let $A \subset \mathbb{R}^d$. Then $m^*(A) \in [0, \infty]$ is defined by

$$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \ell(R_n) : R_n \subset \mathbb{R}^d \text{ closed rectangle, } A \subset \bigcup_{n=1}^{\infty} R_n \right\}$$

Theorem

$m^* : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ is an outer measure, called the **Lebesgue outer measure**.

Lemma

Let m^* be the Lebesgue outer measure and let $R \subset \mathbb{R}^d$ be a closed rectangle with volume $\ell(R)$. Then $m^*(R) = \ell(R)$

2.2.2 Lebesgue measure

Definition *Lebesgue measure*

The **Lebesgue measurable sets** of $\mathbb{R}^d$ are the measurable sets defined by the Lebesgue outer measure m^*, and the **Lebesgue measure** is the restriction of m^* to the Lebesgue measurable sets.
The corresponding complete measure space is denoted by $(\mathbb{R}^d, \mathcal{M}^d, m)$.

Proposition

Every closed rectangle in $\mathbb{R}^d$ is Lebesgue measurable.

Corollary

Every open set in $\mathbb{R}^d$ is Lebesgue measurable.

Corollary

The Lebesgue measure on $\mathbb{R}^d$ is σ-finite.

2.2.3 Behavior under linear transformations

Proposition *Invariance under translation*

Let $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then

$$m^*(A + x) = m^*(A)$$

Hence, $A + x$ is Lebesgue measurable if and only if A is Lebesgue measurable, and in this case,

$$m(A + x) = m(A)$$

Lemma *Invariance under orthogonal transformations*

Let $A \subset \mathbb{R}^d$ and let Q be an orthogonal transformation in $\mathbb{R}^d$. Then

$$m^*(Q(A)) = m^*(A)$$

Hence, $Q(A)$ is Lebesgue measurable if and only if A is Lebesgue measurable, and in this case,

$$m(Q(A)) = m(A)$$

Proposition

Let $A \subset \mathbb{R}^d$ and let T be an invertible linear transformation in $\mathbb{R}^d$ Then

$$m^*(T(A)) = |\det T| m^*(A)$$

Hence, $T(A)$ is Lebesgue measurable if and only if A is Lebesgue measurable, and in this case,

$$m(T(A)) = |\det T| m(A)$$

Corollary

Let T be a non-invertible linear transformation in $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$. Then $T(A)$ is Lebesgue measurable and

$$m(T(A)) = |\det T| m(A) = 0$$

2.2.4 Regularity properties

Lemma

Let $A \subset \mathbb{R}^d$ with $m^*(A) < \infty$. For every $\varepsilon > 0$ there exists an open set $O \subset \mathbb{R}^d$ such that

$$A \subset O \quad \text{and} \quad m^*(A) \leq m(O) < m^*(A) + \varepsilon$$

Lemma

Let $A \subset \mathbb{R}^d$ with $m^*(A) < \infty$. There exists a sequence of open sets $O_n \subset \mathbb{R}^d$ such that

$$A \subset \bigcap_{n=1}^{\infty} O_n \quad \text{and} \quad m^*(A) = m\left(\bigcap_{n=0}^{\infty} O_n\right)$$

Theorem

The Lebesgue measure space $(\mathbb{R}^d, \mathcal{M}^d, m)$ is the completion of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra.

Proposition

Let $A \subset \mathbb{R}^d$. The following are equivalent:

1. A is Lebesgue measurable
2. For all $\varepsilon > 0$ there exists an open set $O \subset \mathbb{R}^d$ such that $A \subset O$ and $m^*(O \setminus A) < \varepsilon$
3. For all $\varepsilon > 0$ there exists a closed set $F \subset \mathbb{R}^d$ such that $F \subset A$ and $m^*(A \setminus F) < \varepsilon$

Corollary

$A \subset \mathbb{R}^d$ is Lebesgue measurable if and only if for all $\varepsilon > 0$ there exist $O, F \subset \mathbb{R}^d$ such that:

- O is open and F is closed
- $F \subset A \subset O$
- $m^*(O \setminus F) < \varepsilon$

2.3 Lebesgue-Stieltjes measure

Proposition *Properties of left and right limits*

Let $F : (a, b) \to \mathbb{R}$ be non-decreasing. Then for $x \in (a, b)$ the left and right limits $F(x-)$ and $F(x+)$ exists, and

1. $\sup\limits_{a<t<x} F(t) = F(x-) \leq F(x) \leq F(x+) = \inf\limits_{x<t<b} F(t)$
2. F has at most countably many discontinuities $x \in (a, b)$, which are jump discontinuities: $F(x-) < F(x+)$
3. If $a < x < y < b$, then $F(x+) \leq F(y-)$

Definition *Relative length of an interval*

Let $F : (a, b] \to \mathbb{R}$ be continuous from the right, i.e. $F(x) = F(x+)$.
We define the **length of** $(a, b]$ **relative to** F by

$$\ell_F(a, b] = F(b) - F(a)$$

Definition *Lebesgue-Stieltjes outer measure on* $\mathbb{R}$

Let $A \subset \mathbb{R}$. Then $(m_F)^*(A) \in [0, \infty]$ is defined by

$$(m_F)^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell_F(I_n) : I_n = (a_n, b_n], A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

Theorem

$(m_F)^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is an outer measure, called the **Lebesgue-Stieltjes outer measure**.

Definition *Pre-measure*

Let $\mathcal{A}$ be a class of subsets of Ω with $\varnothing \in \mathcal{A}$. A function $\mu : \mathcal{A} \to [0, \infty]$ is a **pre-measure** if

1. $\mu(\varnothing) = 0$
2. If $A_n \in \mathcal{A}$ are pairwise disjoint and $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$, then $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$

2.4 Extensions of pre-measures

Definition *Semi-ring*

A collection $\mathcal{S}$ of subsets of Ω is a **semi-ring** if

1. $\varnothing \in \mathcal{S}$
2. if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
3. if $A, B \in \mathcal{S}$, then $A \setminus B$ is a union of pairwise disjoint sets $C_1, \ldots, C_k \in \mathcal{S}$.

Theorem *Carathéodory extension theorem*

Let $\mathcal{S}$ be a semi-ring on Ω and let $\mu : \mathcal{S} \to [0, \infty]$ be a pre-measure.
Then μ has an extension to a measure on $\sigma(\mathcal{S})$, the σ-algebra generated by $\mathcal{S}$.
If the pre-measure on $\mathcal{S}$ is σ-finite, then its extension to $\sigma(\mathcal{S})$ is unique.

3 Measurability of functions

3.1 Measurable functions

Definition *Measurable function*

Let $(\Omega, \mathcal{A})$ and $(\Omega, \mathcal{A}')$ be measurable spaces.
$f : \Omega \to \Omega$ is $(\mathcal{A}, \mathcal{A}')-$**measurable** if $f^{-1}(A') \in \mathcal{A}$ for any $A' \in \mathcal{A}'$.

Proposition

The composition of a $(\mathcal{A}, \mathcal{A}')$-measurable function and a $(\mathcal{A}', \mathcal{A}'')$-measurable function is $(\mathcal{A}, \mathcal{A}'')$-measurable.

Theorem

$$f : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A}') \text{ is measurable} \iff f^{-1}(E') \in \mathcal{A} \text{ for any } E' \in \mathcal{E}' \text{ for some generator } \mathcal{E}' \text{ of } \mathcal{A}'$$

Corollary

Let Y be a topological space.

$$f : (\Omega, \mathcal{A}) \to (Y, \mathcal{B}(Y)) \text{ is measurable} \iff f^{-1}(O) \subset \mathcal{A} \text{ for each open set } O \subset Y$$

3.1.1 Extended real-valued functions

Definition *Extended real line*

$\overline{\mathbb{R}} = [-\infty, \infty]$ is the **extended real line**, equipped with usual ordering, topology and σ-algebra $\overline{\mathcal{B}} := \mathcal{B}(\overline{\mathbb{R}})$

Corollary

$f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable if and only if for all $c \in \mathbb{R}$ one (and hence all) of the following are satisfied:

1. $\{\omega \in \Omega : f(\omega) < c\} \in \mathcal{A}$
2. $\{\omega \in \Omega : f(\omega) \leq c\} \in \mathcal{A}$
3. $\{\omega \in \Omega : f(\omega) > c\} \in \mathcal{A}$
4. $\{\omega \in \Omega : f(\omega) \geq c\} \in \mathcal{A}$

This is also true if we replace $\mathbb{R}$ by the extended real line.

Proposition

$$f = (f_1, f_2) : (\Omega, \mathcal{A}) \to \mathbb{R}^2 \text{ is measurable} \iff f_1 : (\Omega, \mathcal{A}) \to \mathbb{R} \text{ and } f_2 : (\Omega, \mathcal{A}) \to \mathbb{R} \text{ are measurable}$$

This is also true if we replace $\mathbb{R}$ by the extended real line.

Proposition

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and let $A \in \mathcal{A}$. Then the following sets are measurable:

$$\{\omega \in \Omega : f(\omega) < g(\omega)\} \qquad \{\omega \in \Omega : f(\omega) \le g(\omega)\} \qquad \{\omega \in \Omega : f(\omega) = g(\omega)\}$$

Proposition

Let $f, g, f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then the following functions are measurable:

1. $\max(f, g)$ and $\min(f, g)$
2. $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\limsup_{n\in\mathbb{N}} f_n$ and $\liminf_{n\in\mathbb{N}} f_n$
3. $f + g$ (except at points where $f = \pm\infty$ and $g = \mp\infty$)
4. fg and $|f|$
5. cf, $c \in \mathbb{R}$
6. $f/g : \{\omega \in \Omega : g(\omega) \neq 0\} \to \overline{\mathbb{R}}$
7. The pointwise limit of f_n

For the proof, we use the fact that $\overline{\mathcal{B}}$ is generated by the collection of intervals $[-\infty, c]$, $c \in \overline{\mathbb{R}}$.

3.1.2 Probability theory fundamentals

Definition *Probability theory fundamentals*

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, i.e. $(\Omega, \mathcal{A}, \mu)$ is a measure space and $\mu(\Omega) = 1$

- The measurable sets $A \in \mathcal{A}$ are called **events**
- The **probability** of an event $A \in \mathcal{A}$ is defined by $\mu(A)$
- The **conditional probability** of $A \in \mathcal{A}$ given $B \in \mathcal{A}$ is defined by $\mu(A \cap B)/\mu(B)$
- The events $A, B \in \mathcal{A}$ are said to be **independent** if $\mu(A \cap B) = \mu(A)\mu(B)$
- A **random variable** is a measurable function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$

3.2 Approximation by simple functions

Definition *Simple function*

$f : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{B})$ is called **simple** if f is measurable and takes on only a finite number of values.

Theorem

Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable.
Then there exists a sequence of simple functions f_n such that $f_n \to f$ pointwise, and

1. if f is bounded, then the convergence is uniform
2. if $f \ge 0$, then the sequence f_n may be chosen such that $0 \le f_1 \le f_2 \le \dots$

3.3 Properties valid almost everywhere

Definition *Almost everywhere*

A property is said to hold **almost everywhere (a.e.)** on Ω if it holds on a measurable set $A \in \mathcal{A}$ whose complement A^c has measure 0.

Lemma

If $f = g$ almost everywhere and $g = h$ almost everywhere, then $f = h$ almost everywhere.

Lemma

Let $f, g, f_n : (\Omega, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$ be measurable.

1. If f and g are measurable, then
$$f = g \text{ almost everywhere} \iff \mu(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0$$
2. If f_n and f are measurable, then
$$\lim_{n\to\infty} f_n = f \text{ almost everywhere} \iff \mu(\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \neq f(\omega)\}) = 0$$

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space and $f, g, f_n : (\Omega, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$.

1. If f is measurable and $f = g$ almost everywhere, then g is measurable.
2. If f_n is measurable and $\lim_{n\to\infty} f_n = f$ almost everywhere, then f is measurable.

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$ its completion, and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable.

1. f is measurable with respect to $(\Omega, \overline{\mathcal{A}})$
2. There exists a function $g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$, measurable with respect to $\mathcal{A}$ such that
$$\overline{\mu}(\{\omega \in \Omega : g(\omega) \neq f(\omega)\}) = 0$$
i.e. $g = f$ almost everywhere with respect to μ.
Furthermore, g may be constructed such that $g(\omega) = 0$ if $g(\omega) \neq f(\omega)$

4 Integrability of functions

4.1 Integrals of nonnegative functions

Definition *Integral of a nonnegative simple function*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be a simple nonnegative function:
$$f = \sum_{k=1}^{r} \alpha_k \mathbf{1}_{A_k} \qquad r \in \mathbb{N},\ A_k \in \mathcal{A} \text{ pairwise disjoint}$$
Then we define the **integral** of f as:
$$\int_\Omega f \,\mathrm{d}\mu = \sum_{k=1}^{r} \alpha_k \mu(A_k)$$

Proposition

Let $f, g : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be nonnegative simple functions. Let $\alpha > 0$. Then

1. αf and $f + g$ are also nonnegative and simple.
2. $\displaystyle\int_\Omega \alpha f \,\mathrm{d}\mu = \alpha \int_\Omega f \,\mathrm{d}\mu$
3. $\displaystyle\int_\Omega f + g \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu + \int_\Omega g \,\mathrm{d}\mu$
4. $\displaystyle f \geq g \implies \int_\Omega f \,\mathrm{d}\mu \leq \int_\Omega g \,\mathrm{d}\mu$

Corollary

For the definition of the integral, the sets A_k do not need to be disjoint.

Definition *Integral of a nonnegative measurable function*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then the **integral** of f is:

$$\int_\Omega f \,\mathrm{d}\mu = \sup \left\{ \int_\Omega g \,\mathrm{d}\mu : 0 \leq g \leq f,\, g \text{ simple} \right\} \in [0, \infty]$$

Lemma

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Let $f_n : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be a sequence of nonnegative simple functions such that $f_n \uparrow f$. Then

$$\int_\Omega f_n \,\mathrm{d}\mu \uparrow \int_\Omega f_n \,\mathrm{d}\mu$$

Corollary

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$ its completion. Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then f is measurable with respect to $(\Omega, \overline{\mathcal{A}})$, and

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\overline{\mu}$$

Proposition

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be nonnegative measurable functions. Let $\alpha > 0$. Then

1. $\displaystyle\int_\Omega \alpha f \,\mathrm{d}\mu = \alpha \int_\Omega f \,\mathrm{d}\mu$
2. $\displaystyle\int_\Omega f + g \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu + \int_\Omega g \,\mathrm{d}\mu$
3. $\displaystyle f \geq g \implies \int_\Omega f \,\mathrm{d}\mu \leq \int_\Omega g \,\mathrm{d}\mu$

Corollary

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then

$$f = 0 \text{ almost everywhere} \iff \int_\Omega f \,\mathrm{d}\mu = 0 \qquad f = g \text{ almost everywhere} \iff \int_\Omega f \,\mathrm{d}\mu = \int_\Omega g \,\mathrm{d}\mu$$

Corollary

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then

$$\int_\Omega f \,\mathrm{d}\mu < \infty \implies \mu\{\omega \in \Omega : f(\omega) = \infty\} = 0$$

4.2 Integrable functions

Definition *f^+ and f^-*

$$f^+ := \begin{cases} f & \text{if } f \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f^- := \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f & \text{otherwise} \end{cases}$$

Definition *Integral*

Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a measurable function. Then the **integral** of f is defined by

$$\int_\Omega f \, d\mu = \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu$$

if at least one of the integrals on the right-hand side is finite.
f is **integrable** if both integrals on the right-hand side are finite, or equivalently, if $\int_\Omega |f| \, d\mu$ is finite.

Proposition

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable, and let $\alpha \in \mathbb{R}$. Then

1. αf and $f + g$ (when defined) are also integrable
2. $\int_\Omega \alpha f \, d\mu = \alpha \int_\Omega f \, d\mu$
3. $\int_\Omega f + g \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$
4. $f \geq g \implies \int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu$
5. $\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu$

Proposition

Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Assume that f is integrable and $f = g$ almost everywhere. Then g is integrable and

$$\int_\Omega f \, d\mu = \int_\Omega g \, d\mu$$

Corollary

Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable. Then

$$\mu \left(\{ \omega \in \Omega : |f(\omega)| = \infty \} \right) = 0$$

and there exists an integrable function $g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ such that $f = g$ almost everywhere and

$$\int_\Omega f \, d\mu = \int_\Omega g \, d\mu$$

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$ its completion.

1. Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable. Then f is also integrable with respect to $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$, and

$$\int_\Omega f \, d\mu = \int_\Omega f \, d\overline{\mu}$$

2. Let $f : (\Omega, \overline{\mathcal{A}}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable with respect to $\overline{\mu}$. Then $g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ such that $f = g$ w.r.t. $\overline{\mu}$ almost everywhere (as in Theorem 3.4.6*) is integrable with respect to μ, and

$$\int_\Omega g \, d\mu = \int_\Omega f \, d\overline{\mu}$$

* This corresponds with the final theorem of Chapter 3 in these notes.

Definition *Complex integral*

Let $f : (\Omega, \mathcal{A}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ be measurable. Then the function f is said to be **integrable** if the real part f_1 and the imaginary part f_2 are integrable, and the **integral** is defined as

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega f_1 \, \mathrm{d}\mu + i \int_\Omega f_2 \, \mathrm{d}\mu$$

4.3 Integration with respect to the Lebesgue measure

Note

Recall that $(\mathbb{R}^d, \mathcal{M}^d, m)$ is the completion of $(\Omega, \mathcal{B}^d, m_\mathcal{B})$, where $m_\mathcal{B}$ is the Lebesgue measure restricted to $\mathcal{B}^d$. The completion theorem stated at the end of section 4.2 will then also apply to $(\mathbb{R}^d, \mathcal{M}^d, m)$ and $(\Omega, \mathcal{B}^d, m_\mathcal{B})$. Since m and $m_\mathcal{B}$ are equivalent almost everywhere, we will make no distinction between m and $m_\mathcal{B}$.

Theorem *Fundamental theorem of calculus*

Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

1. f is Lebesgue integrable
2. $F(x) = \int_a^x f(t) \, \mathrm{d}t$ is continuously differentiable on $[a, b]$ and $F' = f$.
3. If $G : [a, b] \to \mathbb{R}$ is continuously differentiable with $G' = f$, then $\int_a^b f(t) \, \mathrm{d}t = G(b) - G(a)$

Corollary *Integration by parts*

Let $h, k : [a, b] \to \mathbb{R}$ be measurable. Then

$$\int_a^b h(t)k'(t) = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t) \, \mathrm{d}t$$

Corollary *Substitution rule*

Let $\varphi : [a, b] \to \mathbb{R}$ be continuously differentiable with $\varphi' > 0$, and let $f : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be continuous. Then $(f \circ \varphi)\varphi'$ is Lebesgue integrable on $[a, b]$ and

$$\int_a^b (f \circ \varphi)(x)\varphi'(x) \, \mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(y) \, \mathrm{d}y$$

4.4 Convergence theorems

Theorem *Monotone convergence theorem*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be nonnegative measurable functions such that

$$f_n(\omega) \to f(\omega) \qquad f_n(\omega) \leq f_{n+1}(\omega) \qquad \omega \in \Omega$$

Then

$$\lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu$$

Lemma *Lebesgue integration on the real line*

Let $f : (a, b] \to \mathbb{R}$ or $f : [a, b) \to \mathbb{R}$ be nonnegative and continuous with primitive F. Then, respectively,

$$\int_{(a,b]} f \, \mathrm{d}m = \lim_{\alpha \searrow a} [F(b) - F(\alpha)] \qquad \text{or} \qquad \int_{[a,b)} f \, \mathrm{d}m = \lim_{\beta \nearrow b} [F(\beta) - F(a)]$$

Theorem *Fatou's lemma*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be nonnegative and measurable. Then

$$\int_\Omega \left(\liminf_{n\to\infty} f_n \, \mathrm{d}\mu \right) \le \liminf_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu$$

Theorem *Dominated convergence theorem*

Let $f_n, f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Let $g : \Omega \to [0, \infty]$ be an integrable function such that

$$f_n(\omega) \to f(\omega) \qquad |f_n(\omega)| \le g(\omega) \qquad \text{for almost all } \omega \in \Omega$$

Then f_n, f are integrable and

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \, \mathrm{d}\mu = 0 \qquad \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu$$

Proposition

Let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then

$$\int_\Omega \left(\sum_{n=1}^\infty f_n \right) \mathrm{d}\mu = \sum_{n=1}^\infty \int_\Omega f_n \, \mathrm{d}\mu$$

Corollary

Let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. If $\Omega_n \in \mathcal{A}$ are disjoint and $\bigcup_{n=1}^\infty \Omega_n = \Omega$, then

$$\int_\Omega f \, \mathrm{d}\mu = \sum_{n=1}^\infty \int_{\Omega_n} f \, \mathrm{d}\mu$$

Theorem

Let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable functions such that

$$\sum_{n=1}^\infty \int_\Omega f_n \, \mathrm{d}\mu < \infty$$

Then the series $\sum_{n=1}^\infty f_n$ converges almost everywhere to an integrable extended real-valued function and

$$\int_\Omega \left(\sum_{n=1}^\infty f_n \right) \mathrm{d}\mu = \sum_{n=1}^\infty \int_\Omega f_n \, \mathrm{d}\mu$$

4.5 Parameter-dependent integrals

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space and $I \subset \mathbb{R}$ an open interval with $x_0 \in I$.
Let $f : I \times \Omega \to \mathbb{R}$ be a real-valued function such that:

1. for every $x \in I$ the function $\omega \mapsto f(x, \omega)$ is integrable.
2. for almost all $\omega \in \Omega$ the function $x \mapsto f(x, \omega)$ is continuous at $x_0 \in I$

Define the following real-valued function:

$$F : I \to \mathbb{R} \qquad F(x) = \int_\Omega f(x, \omega)\,\mathrm{d}\mu(\omega)$$

- If there exists an integrable real-valued function $g : \Omega \to \mathbb{R}$ such that

$$|f(w, \omega)| \leq |g(\omega)| \qquad \text{for all } x \in I \qquad \text{almost everywhere on } \Omega$$

  then F is continuous at $x_0 \in I$.
- If there exists an integrable real-valued function $g : \Omega \to \mathbb{R}$ such that

$$\left|\frac{\partial}{\partial x} f(w, \omega)\right| \leq |g(\omega)| \qquad \text{for all } x \in I \qquad \text{almost everywhere on } \Omega$$

  then F is differentiable at $x_0 \in I$, and its derivative at x_0 is

$$F'(x_0) = \int_\Omega \left.\frac{\partial}{\partial x}\right|_{x_0} f(x, \omega)\,\mathrm{d}\mu(\omega)$$

4.6 Densities and transformations of measures

Theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, let $h : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be nonnegative and measurable, let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable, and define λ by

$$\lambda(E) \int_E h\,\mathrm{d}\mu \qquad E \in \mathcal{A}$$

λ is a measure on $\mathcal{A}$, and the following statements hold:

1. If f is nonnegative, then
$$\int_\Omega f\,\mathrm{d}\lambda = \int_\Omega fh\,\mathrm{d}\mu \tag{$*$}$$
2. $$f \text{ integrable with respect to } \lambda \iff fh \text{ integrable with respect to } \mu \implies (*) \text{ holds}$$

Note: h is sometimes called the **density** of λ with respect to μ.

Theorem

Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, and let $\varphi : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ and $f : (\Omega', \mathcal{A}') \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. If μ is a measure on $(\Omega, \mathcal{A})$, then v defined by

$$v(B) = \mu(\varphi^{-1}(B)) \qquad B \in \mathcal{A}'$$

is a measure on $(\Omega', \mathcal{A}')$, and the following statements hold:

1. If f is nonnegative, then
$$\int_{\Omega'} f\,\mathrm{d}v = \int_\Omega (f \circ \varphi)\,\mathrm{d}\mu \tag{$**$}$$
2. $$f \text{ integrable with respect to } v \iff f \circ \varphi \text{ integrable with respect to } \mu \implies (**) \text{ holds}$$

Note: v is called the **induced measure**, and is sometimes denoted by $v = \mu \circ \varphi^{-1}$.

Definition *Measure preserving mapping*

Let μ be a measure on $(\Omega, \mathcal{A})$. The mapping $\varphi : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A})$ is said to be **measure preserving** if

$$\mu(B) = \mu(\varphi^{-1}(B)) \qquad \text{for all } B \in \mathcal{A}$$

5 Product measures

5.1 Construction of product measures

Definition *Product σ-algebra*

Let $\mathcal{A}_1, \mathcal{A}_2$ be σ-algebras on Ω_1, Ω_2 respectively. The σ-algebra generated by

$$\mathcal{G} = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$$

is called the **product σ-algebra** $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ on $\Omega_1 \times \Omega_2$.

Definition *Sections*

Let A be a subset of $\Omega = \Omega_1 \times \Omega_2$. For $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the **sections** A_{ω_1} and A^{ω_2} are defined as

$$A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\} \subset \Omega_2 \qquad A^{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in A\} \subset \Omega_1$$

Proposition

Let $A = \mathcal{A}_1 \otimes \mathcal{A}_2$. Then

$$\omega_1 \in \Omega_1 \implies A_{\omega_1} \in \mathcal{A}_2 \qquad \omega_2 \in \Omega_2 \implies A^{\omega_2} \in \mathcal{A}_1$$

Definition *Function sections*

Consider the function $f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$. For $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the **sections** f_{ω_1} and f^{ω_2} are defined as

$$f_{\omega_1} = f(\omega_1, \omega_2) \quad \omega_2 \in \Omega_2 \qquad f^{\omega_2} = f(\omega_1, \omega_2) \quad \omega_1 \in \Omega_1$$

Proposition

Let $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then

1. For $\omega_1 \in \Omega_1$, the function $f_{\omega_1} : (\Omega_2, \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable.
2. For $\omega_2 \in \Omega_2$, the function $f^{\omega_2} : (\Omega_1, \mathcal{A}_1) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable.

Proposition

Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces and let $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then

1. The function $\omega_1 \mapsto \mu_2(A_{\omega_1})$ is measurable with respect to $\mathcal{A}_1$. The function $\omega_2 \mapsto \mu_1(A^{\omega_2})$ is measurable with respect to $\mathcal{A}_2$.

Theorem *Product measure*

Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces and let $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then

$$\mu(A) = \int_{\Omega_1} \mu_2(A_{\omega_1}) \mu_1(\mathrm{d}\omega_1) = \int_{\Omega_2} \mu_1(A^{\omega_2}) \mu_2(\mathrm{d}\omega_2)$$

defines a σ-finite measure $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover, μ is the only measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \qquad A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$$

Note

The product of complete measures is not necessarily complete.

5.2 Fubini-Tonellini theorems

Theorem *Fubini-Tonellini theorem (nonnegative functions)*

Assume μ_1 and μ_2 are σ-finite. Assume that $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable and nonnegative. Then,

1. The function $\omega_1 \mapsto \int_{\Omega_2} f_{\omega_1} \, \mathrm{d}\mu_2$ is nonnegative and $\mathcal{A}_1$-measurable.
2. The function $\omega_2 \mapsto \int_{\Omega_1} f^{\omega_2} \, \mathrm{d}\mu_1$ is nonnegative and $\mathcal{A}_2$-measurable.

Moreover,

$$\int_{\Omega_1 \times \Omega_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f_{\omega_1} \, \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1(\omega_1) = \int_{\Omega_2} \left(\int_{\Omega_1} f^{\omega_2} \, \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2(\omega_2) \qquad \text{in } [0, \infty]$$

Corollary *Fubini-Tonellini theorem (absolute value)*

Assume μ_1 and μ_2 are σ-finite. Assume that $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable. Then,

$$\int_{\Omega_1 \times \Omega_2} |f| \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} |f_{\omega_1}| \, \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1(\omega_1) = \int_{\Omega_2} \left(\int_{\Omega_1} |f^{\omega_2}| \, \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2(\omega_2)$$

In particular, the integrals are simultaneously finite.

Theorem *Fubini-Tonellini theorem (integrable functions)*

Assume μ_1 and μ_2 are σ-finite. Assume that $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable.
Define the following sets:

$F_1 = \{\omega_1 \in \Omega_1 : f_{\omega_1} \text{ not integrable with respect to } \mu_2\}$ $\quad F_2 = \{\omega_2 \in \Omega_2 : f^{\omega_2} \text{ not integrable with respect to } \mu_1\}$

F_1 has μ_1-measure 0 and F_2 has μ_2-measure 0. Moreover,

$$\int_{\Omega_1 \times \Omega_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1 \setminus F_1} \left(\int_{\Omega_2} f_{\omega_1} \, \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1(\omega_1) = \int_{\Omega_2 \setminus F_2} \left(\int_{\Omega_1} f^{\omega_2} \, \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2(\omega_2)$$

5.3 Product measures on Euclidean spaces

Proposition *Product of Borel σ-algebras*

Let $\mathcal{B}^p$ and $\mathcal{B}^q$ be Borel σ-algebras equipped with the Lebesgue measure m_p, m_q.

$$\mathcal{B}^p \otimes \mathcal{B}^q = \mathcal{B}^{p+q} \qquad m_p \otimes m_q = m_{p+q} \text{ (well-defined)}$$

Corollary *Fubini-Tonellini theorem (elementary version)*

Let $K = K_p \times K_q \subset \mathbb{R}^p \times \mathbb{R}^q$ be compact. Let $f : K \to \mathbb{R}$ be continuous and let $A \subset K$ be Borel measurable. Then

$$\int_A f \, \mathrm{d}m_{p+q} = \int_{K_p} \left(\int_{K_q} (f \mathbf{1}_A)(x, y) \, \mathrm{d}m_q(y) \right) \mathrm{d}m_p(x) = \int_{K_q} \left(\int_{K_p} (f \mathbf{1}_A)(x, y) \, \mathrm{d}m_p(x) \right) \mathrm{d}m_q(y)$$

Proposition *Product of Lebesgue σ-algebras*

Let $\mathcal{M}^p$ and $\mathcal{M}^q$ be Lebesgue σ-algebras equipped with the Lebesgue measure m_p, m_q.

$$\overline{\mathcal{M}^p \otimes \mathcal{M}^q} = \mathcal{M}^{p+q} \qquad \overline{m_p \otimes m_q} = m_{p+q} \text{ (well-defined)}$$

5.4 Convolutions

Lemma

Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable. Then the following functions are Lebesgue measurable:

$$h, k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \qquad h(x, y) = f(x)g(y) \qquad k(x, y) = f(x - y)g(y)$$

Definition *Convolution*

Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable. The **convolution product** of f and g is defined as

$$h(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\,\mathrm{d}m(y) \qquad \text{wherever } f(x - y)g(y) \text{ is integrable}$$

The function h, denoted $f * g$, is the **convolution** of f and g.

Theorem

Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue integrable. Then

$$\int_{\mathbb{R}^d} |f(x - y)g(y)|\,\mathrm{d}m(y) < \infty \qquad \text{for almost all } x \in \mathbb{R}^d$$

For all $x \in \mathbb{R}^d$ where this integral is finite, define $h(x)$ as in the definition of convolution. Then

$$\int_{\mathbb{R}}^{d} |h(x)|\,\mathrm{d}m(x) \leq \left(\int_{\mathbb{R}}^{d} |f(x)|\,\mathrm{d}m(x)\right)\left(\int_{\mathbb{R}}^{d} |g(x)|\,\mathrm{d}m(x)\right)$$

6 Spaces of integrable functions

6.1 Normed linear spaces

Definition *Semi-norm and norm*

A **semi-norm** on a linear space $\mathcal{X}$ over $\mathbb{C}$ is a function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ satisfying:

1. $\|h\| \geq 0$ for all $h \in \mathcal{X}$
2. $\|\lambda h\| = |\lambda|\|h\|$ for all $h \in \mathcal{X}$ and $\lambda \in \mathbb{C}$
3. $\|h + k\| \leq \|h\| + \|k\|$ for all $h, k \in \mathcal{X}$

A semi-norm is a **norm** if $\|h\| = 0 \iff h = 0$. A linear space with a norm is called a **normed linear space**.

Definition *Complete linear space*

A **Cauchy sequence** is a sequence h_n such that

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies \|h_n - h_m\| < \varepsilon$$

A sequence h_n **converges** to h if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } n \geq N \implies \|h_n - h\| < \varepsilon$$

A semi-normed linear space is **complete** if every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$.
A complete normed linear space is called a **Banach space**.

Lemma

A Cauchy sequence converges if it has a convergent subsequence.

Definition *Set of neutral elements*

The set of **neutral elements** in a semi-normed linear space $\mathcal{X}$ is:

$$\mathcal{N} = \{h \in \mathcal{X} : \|h\| = 0\}$$

Lemma

$\mathcal{N}$ is a closed linear subspace of $\mathcal{X}$.

Definition *Convergent series*

A series $\sum_{n=1}^{\infty} f_n$ is **convergent** if its sequence of partial sums converges, and **absolutely convergent** if $\sum_{n=1}^{\infty} \|f_n\| < \infty$.

Proposition

$$\mathcal{X} \text{ is complete} \iff \text{every absolutely convergent series in } \mathcal{X} \text{ converges in } \mathcal{X}$$

Definition $\mathcal{X}/\mathcal{N}$

For $h, k \in \mathcal{X}$ define $h \sim k$ if $h - k \in \mathcal{N}$. The set of corresponding equivalence classes $[h]$ is denoted $\mathcal{X}/\mathcal{N}$. $\mathcal{X}/\mathcal{N}$ provided with the norm $\|[h]\| = \|h\|$ is a normed linear space.

Lemma

If $\mathbb{X}$ is a complete normed linear space, then $\mathbb{X}/\mathbb{N}$ is a Banach space.

6.2 Hölder and Minkowski inequalities

Lemma

Let $\alpha, \beta \in \mathbb{R}$ be nonnegative and let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q} \qquad \left(\frac{\alpha+\beta}{2}\right)^p \leq \frac{1}{2}(\alpha^p + \beta^p)$$

Definition $\mathcal{L}^p$

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \geq \infty$.

- For $p < \infty$ the space $\mathcal{L}^p(\Omega)$ is the set of all measurable complex-valued functions f for which

  $$\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$$

  Furthermore, we define

  $$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p}$$

- The space $\mathcal{L}^\infty$ is the set of all measurable complex-valued functions f for which

  $$\text{there exists } c \geq 0 \text{ such that } |f(\omega)| \leq c \text{ almost everywhere}$$

  Furthermore, we define

  $$\|f\|_\infty = \inf\{c \geq 0 : |f(\omega)| \leq c \text{ almost everywhere}\}$$

Lemma

$\mathcal{L}^p$ is a linear space for all $1 \leq p \leq \infty$.

Theorem *Hölder's inequality*

Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If $f \in \mathcal{L}^p(\Omega)$ and $g \in \mathcal{L}^q(\Omega)$, then $fg \in \mathcal{L}^1(\Omega)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

The case $p = 2, q = 2$ is the **Cauchy-Schwarz inequality**.

Corollary

Let $1 \le p \le \infty$ and assume $\mu(\Omega) = 1$. If $f \in \mathcal{L}^p(\Omega)$, then $f \in \mathcal{L}^p(\Omega)$ and

$$\|f\|_1 \le \|f\|_p$$

Theorem *Minkowski's inequality*

Let $1 \le p \le \infty$. For $f, g \in \mathcal{L}^p$, we have

$$\|f+g\|_p \le \|f\|_p + \|g\| + p$$

Corollary

$\mathcal{L}^p(\Omega)$ with $\|\cdot\|_p$ is a semi-normed linear space.
A function $f \in \mathcal{L}^p(\Omega)$ has $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.

6.3 Completeness

Theorem

Let $1 \le p \le \infty$ and $f_n \in \mathcal{L}^p$.

$$\sum_{k=1}^{\infty} f_k \text{ converges absolutely} \implies \begin{cases} \sum_{k=1}^{\infty} f_k \text{ converges in } \mathcal{L}^p(\Omega) \\ \sum_{k=1}^{\infty} f_k \text{ converges pointwise almost everywhere} \end{cases}$$

Theorem

Let $1 \le p \le \infty$. The semi-normed linear space $\mathcal{L}^p(\Omega)$ is complete.

Corollary

Let $1 \le p \le \infty$.
Every Cauchy sequence in $\mathcal{L}^p(\Omega)$ contains a subsequence that converges pointwise almost everywhere.

Definition L^p

$$L^p(\Omega, \mu) = \mathcal{L}^p(\Omega, \mu)/\mathcal{N}(\mu)$$

Natural embedding

Let $\overline{\mu}$ be the completion of μ and consider the equivalence classes $[f]_\mu \in L^p(\Omega, \mu)$ and $[f]_{\overline{\mu}} \in L^p(\Omega, \mu)$
Then the **natural embedding** $[f]_\mu \mapsto [f]_{\overline{\mu}}$ is a well-defined linear mapping from $L^p(\Omega, \mu)$ to $L^p(\Omega, \overline{\mu})$.
Moreover, it preserves the norm:

$$\|[f]_\mu\|_p = \|[f]_{\overline{\mu}}\|_p$$

Theorem

The natural embedding is surjective.

Theorem

Let $1 \le p \le \infty$. The simple functions which belong to $\mathcal{L}^p(\Omega)$ are dense in $\mathcal{L}^p(\Omega)$.

Definition *Compact support*

A function $f : \mathbb{R}^d \to \mathbb{C}$ has **compact support** if the closure of $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ is bounded.
The space $C_c(\mathbb{R}^d)$ is the collection of all functions $f : \mathbb{C} \to \mathbb{R}^d$ that are continuous and have compact support.

Theorem

The space $C_c(\mathbb{R}^d)$ is a dense subset of $\mathcal{L}^d(\mathbb{R}^d)$ for $1 \le p < \infty$.

Corollary

Let $f \in \mathcal{L}^p(\mathbb{R}^d)$ with $1 \leq p < \infty$. Then

$$\lim_{x \to 0} \int_{\mathbb{R}^d} |f(x+t) - f(t)|^p \, \mathrm{d}m(t) = 0$$

7 Decomposition of measures

7.1 Bounded linear functionals on Hilbert spaces

Definition *Bounded linear functional*

Let $\mathcal{X}$ be a normed linear space over $\mathbb{C}$. A **linear functional** is a linear function $\mathcal{X} \to \mathbb{C}$.
We define the **norm of a linear functional** $F : \mathcal{X} \to \mathbb{C}$ by

$$\|F\| = \sup \left\{ \frac{|Fh|}{\|h\|} : h \in \mathcal{X}, h \neq 0 \right\}$$

F is **bounded** if $\|F\| < \infty$.

Definition *Dual space*

The **dual space** of $\mathcal{X}$ is defined by

$$\mathcal{X}' = \{F : \mathcal{X} \to \mathbb{C} : F \text{ is bounded and linear}\}$$

Theorem

The dual of any normed linear space is a Banach space.

Definition *Hilbert space*

A vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space**.
Inner products induce the following norm:

$$\|h\| = \sqrt{\langle h, h \rangle}$$

A complete inner product space is called a **Hilbert space.**

Theorem *Cauchy-Schwarz inequality*

$$|\langle h, k \rangle| \leq \|h\| \|k\|$$

Definition F_k

$$F_k : \mathcal{X} \to \mathbb{C} \qquad h \mapsto \langle h, k \rangle$$

Proposition

Let $\mathcal{X}$ be an inner product space. Then $F_k \in \mathcal{X}'$.

Theorem *Riesz representation theorem*

Let $\mathbb{X}$ be a Hilbert space and let $F \in \mathcal{X}'$. Then there exists a unique element $k \in \mathcal{X}$ such that $F = F_k$.

7.2 Domination

Definition *Domination*

Let ν, μ be measures on $(\Omega, \mathcal{A})$. Then ν is **dominated** by μ if

$$\nu(A) \leq \mu(A) \text{ for all } A \in \mathcal{A}$$

Theorem *Radon-Nikodym derivative (domination case)*

Let μ be a finite measure on $(\Omega, \mathcal{A})$ and let ν be a measure on $(\Omega, \mathcal{A})$ which is dominated by μ. Then there exists a μ-measurable real-valued function h with $0 \leq h \leq 1$ almost everywhere with respect to μ, such that

$$\nu(A) = \int_A h \, \mathrm{d}\mu \text{ for all } A \in \mathcal{A}$$

This function h is uniquely determined up to sets of μ-measure 0.

7.3 Absolutely continuous measures

Definition *Sum of measures*

Let λ_1, λ_2 be two measures on $(\Omega, \mathcal{A})$. Their **sum** is defined by

$$(\lambda_1 + \lambda_2)(A) = \lambda_1(A) + \lambda_2(A)$$

Definition *Absolutely continuous measure*

Let λ and μ be measures on $(\Omega, \mathcal{A})$. Then λ is **absolutely continuous** with respect to μ, denoted $\lambda \ll \mu$, if

$$A \in \mathcal{A} \text{ and } \mu(A) = 0 \implies \lambda(A) = 0$$

Lemma

Let $\lambda_1, \lambda_2, \mu$ be measures on $(\Omega, \mathcal{A})$. Then

$$\lambda_1 \ll \mu \text{ and } \lambda_2 \ll \mu \implies \lambda_1 + \lambda_2 \ll \mu$$

Theorem *Radon-Nikodym derivative*

Let λ and μ be σ-finite measures on $(\Omega, \mathcal{A})$ with $\lambda \ll \mu$.
Then there exists a μ-measurable extended real-valued function h such that

$$\lambda(A) = \int_A h \, \mathrm{d}\mu \qquad A \in \mathcal{A}$$

The function h is uniquely determined up to sets of μ-measure 0.

7.4 Lebesgue decomposition

Definition *Concentrated measure*

A measure λ on $(\Omega, \mathcal{A})$ is **concentrated** on a set $S \in \mathcal{A}$ if $\lambda(S^c) = 0$.

Lemma

Let λ be a measure on $(\Omega, \mathcal{A})$ and let $S \in \mathcal{A}$. The following are equivalent:

1. The measure λ is concentrated on S.
2. For all $A \in \mathcal{A}$ we have $A \subset S^c \implies \lambda(A) = 0$
3. For all $A \in \mathcal{A}$ we have $\lambda(A) = \lambda(A \cap S)$

Definition *Mutually singular measures*

Let λ_1, λ_2 be measures on $(\Omega_1, \mathcal{A})$. Then λ_1, λ_2 are **mutually singular**, denoted $\lambda_1 \perp \lambda_2$,
if λ_1 is concentrated on $S_1 \in \mathcal{A}$, λ_2 is concentrated on $S_2 \in \mathcal{A}$, and $S_1 \cap S_2 = \varnothing$.

Lemma

Let λ_1, λ_2 be measures on $(\Omega, \mathcal{A})$. The following are equivalent:

1. λ_1 and λ_2 are mutually singular.
2. There exists a set $N \in \mathcal{A}$ such that λ_1 is concentrated on N and λ_2 is concentrated on N^c.

Lemma

Let $\lambda_1, \lambda_2, \lambda, \mu$ be measures on $(\Omega, \mathcal{A})$.

1. $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu \implies \lambda_1 + \lambda_2 \perp \mu$
2. $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu \implies \lambda_1 \perp \lambda_2$
3. $\lambda \ll \mu$ and $\lambda \perp \mu \implies \lambda = 0$

Theorem *Lebesgue decomposition*

Let λ and μ be σ-finite measures on $(\Omega, \mathcal{A})$. There exists a unique pair of measures λ_a and λ_s such that

$$\lambda = \lambda_a + \lambda_s \qquad \lambda_a \ll \mu \qquad \lambda_s \perp \mu$$

and in addition, $\lambda_a \perp \lambda_s$.

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